Behavioural And Theoretical Support for Ranking Theory as an Alternative Model of Human Uncertainty Representation

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What is "Ranking Theory"?

- A normative model that quantifies degrees of belief and disbelief using natural numbers and infinity, providing a framework for representing human belief systems[1].

Study Objective:

- To investigate the empirical adequacy of ranking theory in measuring belief and disbelief across various contexts and scenarios (E1:E5), particularly in comparison to a traditional probabilistic model and decision-making framework.

Metrics of Belief and Disbelief

A grading of **disbelief** (or surprise) expressed by ***negative*** *ranking function*, κ : *(scale from 0 to ∞)*[1]

$$\kappa(A) = 0 : A \text{ is not disbelieved (not surprising)} \quad (1)$$

$$\kappa(A) > 0 : A \text{ is disbelieved (surprising)} \quad (2)$$

$$\kappa(A) = \infty : A \text{ is considered impossible} \quad (3)$$

$$\kappa(A) = 0 \text{ or } \kappa(\bar{A}) = 0 : \text{[the law of negation]} \quad (4)$$

A grading of **belief** expressed by ***positive*** *ranking function*, β : *(scale from 0 to ∞)*[1]

$$\beta(A) = \kappa(\bar{A}) : \text{Belief in } A \text{ equals disbelief in not-}A \quad (5)$$

Integrating *positive* and *negative* ranks into a ***two-sided*** ranking function τ expresses **belief and disbelief** at once: *(scale from - ∞ to ∞)*[1]

$$\tau(A) = \beta(A) - \kappa(A) = \kappa(\bar{A}) - \kappa(A) \quad (6)$$

Probability-Rank Translation

Probability to *negative* ranking function[2]:

$$\kappa(A) = \log_b P(A) - \log_b \max_{S \in \Omega} P(S),\ b \in (0,1) \quad (7)$$

Probability to *two-sided* ranking function[2]:

$$\tau(A) = \log_b \left(\frac{1 - P(A)}{P(A)}\right),\ b \in (0,1) \quad (8)$$

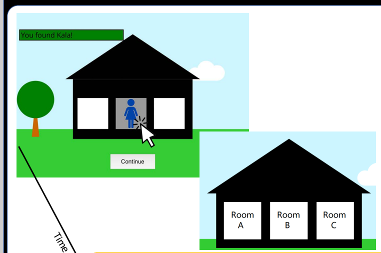
E1:E3 Methods – Reasoning Under Uncertainty

Grade your degrees of **disbelief:**
*(reminder: **0** for **not surprising**, value **greater than 0** for **surprising**, with infinity for impossible)*
Real Estate Investing:
Suppose you buy a property in Toronto, Ontario, today. How **surprised** would you be if the property's value __________ 5 years from today?

increased | decreased | remained the same
*did **not** increase | *did **not** decrease | *did **not** remain the same

*(**Experiment 3 (E3) Specific Questions:***)*

E4 Methods – Dynamic Learning Environment

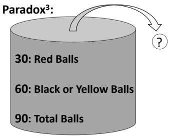


Grade your degrees of *disbelief*:
*(reminder: **0** for **not surprising**, value **greater than 0** for **surprising**, with infinity for impossible)*
How **surprised** would you be if Kala was hiding in ______?

Room A | Room B | Room C
not Room A | **not** Room B | **not** Room C

E5 Methods – Decision Making Task

The Ellsberg One-Urn Paradox[3]:



Grade your degrees of *disbelief*:
*(reminder: **0** for **not surprising**, value **greater than 0** for **surprising**, with infinity for impossible)*
How **surprised** would you be if the ball you drew was ______?

red | black
red **or** yellow | black **or** yellow

Please **choose** one of the gambles for **each scenario**. *(repeated x4)*

Scenario 1:
A: Receive $100 if Red, otherwise nothing
B: Receive $100 if Black, otherwise nothing

Scenario 2:
C: Receive $100 if Red or Yellow, otherwise nothing
D: Receive $100 if Black or Yellow, otherwise nothing

Time

People Can Grade Degrees of Belief and Disbelief Using Ranking Functions, Adhering to the Law of Negation

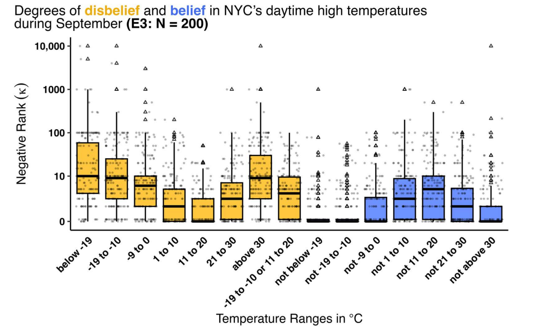


Figure 1. Participants' numerical ratings of both belief (positive ranks, Eq. 5) and disbelief (negative ranks, Eq. 1-3) regarding various temperature ranges, assessed during a questionnaire without known objective probabilities (E1-E3). Both rank types consistently aligned with societal and cultural expectations. Notably, most participants adhered to the law of negation (Eq. 4), demonstrating logical consistency by not simultaneously disbelieving both a proposition and its negation (e.g., "11 to 20°C" and "not 11 to 20°C"). This highlights their capacity to navigate logical constraints in belief assessments.

Contextual Variations in Subjective Disbelief Thresholds

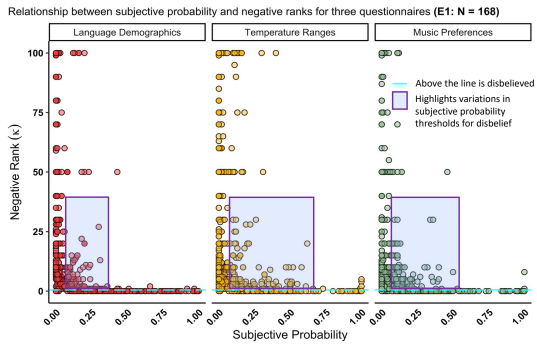


Figure 2. The blue line represents the disbelief threshold (Eq. 2); values below it indicate that the proposition is not disbelieved and may represent some degree of belief (Eq. 1). For propositions not disbelieved, participants provided the entire range of subjective probabilities from 0 to 1. The purple box highlights varying subjective probability thresholds for disbelief, indicating context dependence. This suggests that subjective probability alone may not be sufficient to explain degrees of belief and disbelief.

Distinctive Nature of Negative Ranks vs. Subjective Probabilities

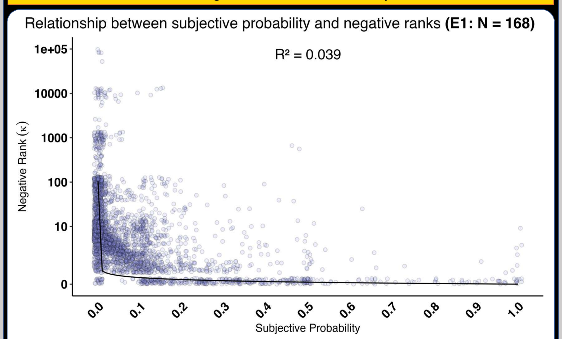


Figure 3. This plot illustrates the complex relationship between subjective probabilities and negative ranks (disbelief). In broader contexts of uncertainty (E1-E3), the mathematical relationship is less evident. The R^2 value of 0.039 indicates a weak correlation, highlighting that negative ranks are not merely logarithmic transformations of probabilities (Eq. 7). Instead, they reflect a distinct measure of uncertainty, capturing varying degrees of disbelief that traditional probabilistic models may overlook. However, in a probabilistic learning task (E4), where frequencies are observed and tracked based on the number of events, the logarithmic function explained 55.4% of the variance. In such scenarios, belief formation is influenced by empirical probabilities, though its interpretation fundamentally differs from probabilities.

Two-Sided Ranks Capture the Full Spectrum of Beliefs

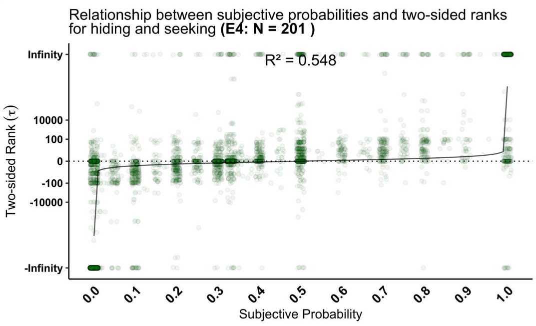


Figure 4. Incorporating both positive and negative ranks allowed for a neutral stance and captured the full belief spectrum, which traditional probabilistic assessments struggle to achieve. In a probabilistic learning task (E4), a log-odds relationship (Eq. 8) explained 54.8% of the variance, showing that strong disbelief corresponds with very low probabilities, and strong belief with high probabilities. A neutral stance (neither disbelieved nor believed) spanned a wide range of probabilities [0,1]. However, in the broader context of reasoning under uncertainty (E3), this log-odds relationship only captured 6.4% of the variance.

Correlation Between Two-Sided Ranks and Objective Probabilities

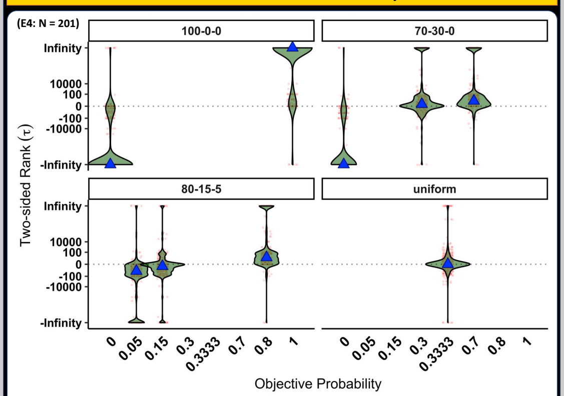


Figure 5. This violin plot illustrates the relationship between objective probabilities and two-sided ranks across various probability distributions. The range extends from negative infinity (representing impossibility) to infinity (representing certainty). Negative values indicate disbelief, while positive values signify belief. The medians, marked by triangles, show central tendencies within each distribution. A strong positive monotonic relationship was observed ($r = 0.77$, $p < .001$), indicating that as objective probabilities increase, so do the two-sided ranks.

Degrees of Disbelief More Closely Align with Objective Probability Distribution than Subjective Probability

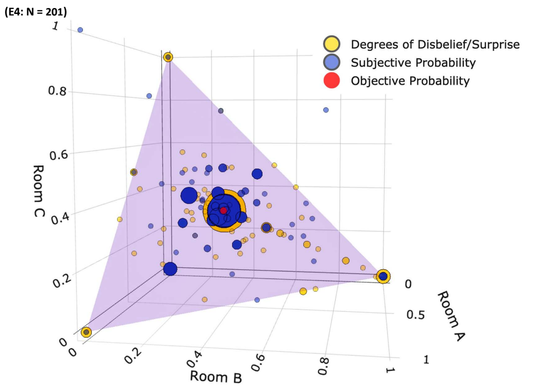


Figure 6. A probability simplex ($\Delta n-1$) shows participants' subjective probabilities (blue dots) and disbelief rankings (yellow dots) against a fixed objective probability [1/3, 1/3, 1/3] in red. The dispersed blue dots illustrate varied subjective responses (**34.33% agreement**), while the clustered yellow dots at the center reflect more consistent evaluations of the objective probability (**58.21% agreement**). Larger dots indicate more common responses, demonstrating that rankings provide a clearer view of uncertainty structure than subjective probabilities.

Ambiguity Aversion Diminishes via Negative Ranks

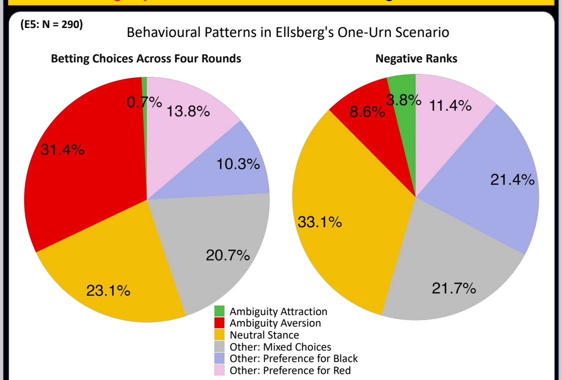


Figure 7. When participants used negative ranking functions, the proportion showing ambiguity aversion (Ellsberg paradox) dropped to 8.6%, compared to 31.4% with betting choices. This indicates that ranking theory offers a less biased and purer measure of belief under uncertainty.

Inaccuracy of Choice-Based Methods in Inferring Belief

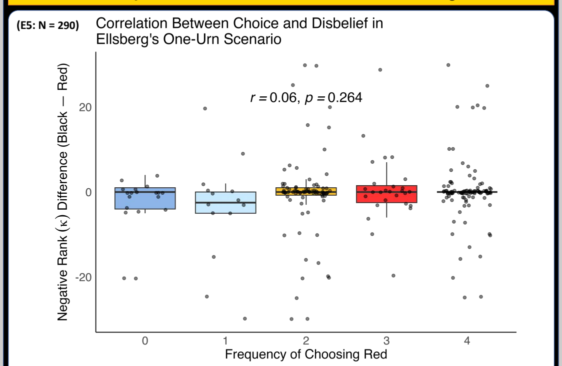


Figure 8. No correlation was found between participants' degrees of disbelief expressed through ranking functions and their betting choices. This challenges the reliability of choice-based methods in inferring beliefs and highlights the complexity of belief formation. Choices with costs and benefits do not reflect underlying beliefs, while ranks of disbelief provide a purer measure.

1. Spohn, W. (2009). A survey of ranking theory. In *Degrees of belief* (pp. 185-228). Dordrecht: Springer Netherlands.
2. Raidl, E., & Skovgaard-Olsen, N. (2017). Bridging ranking theory and the stability theory of belief. *Journal of Philosophical Logic*, *46*(6), 577-609.
3. Ellsberg, D. (1961). The crude analysis of strategy choices. *The American Economic Review*, *51*(2), 472-478.

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Behavioural and theoretical support for ranking theory as an alternative model of human uncertainty representation

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Abstract

Measuring and quantifying degrees of belief poses a fundamental challenge, prompting an exploration into how humans navigate uncertainty. This study challenges the conventional use of probability theory and investigates ranking theory as a viable alternative model. Across the initial three experiments (N = 168; N = 63; N = 200), participants consistently utilized negative ranking functions to express disbelief, revealing a robust pattern across diverse contexts. Notably, a logarithmic relationship emerged between subjective probability and negative ranks (degree of disbelief), highlighting the granularity of ranking functions. Experiment 3 introduced positive ranks, illustrating a log-odds relationship between subjective probability and two-sided ranks (degree of disbelief and belief), providing a detailed depiction of the full spectrum of beliefs. In Experiment 4 (N = 201), examining ranks and subjective probability in a learning task revealed that disbelief via negative ranking functions more accurately represented the objective probability distribution than subjective probability. Lastly, Experiment 5 (N = 291) addressed decision-making under uncertainty through the Ellsberg paradox, uncovering how ranking theory not only resolved contradictions with expected utility theory but also eliminated the paradoxical nature of the Ellsberg scenario. This study advances our understanding of human uncertainty and supports ranking theory as a compelling alternative.

Keywords: ranking theory; belief representation; reasoning under uncertainty; decision-making; Ellsberg paradox.

Introduction

How do humans represent beliefs and uncertainties? This fundamental question in cognitive science serves as the foundation for reasoning, learning, and decision-making. Despite its significance, the mechanisms of human uncertainty representation remain unclear. Probability theory is a widely used framework for representing beliefs, where each state of the world is assigned a probability. This model guides decision-making processes and actions based on established probability laws (Sanborn & Chater, 2016). However, there is an ongoing debate about whether this approach can fully capture the inherent nature of belief and uncertainty (Galavotti, 2017). In the probability theory framework, degrees of belief adhere to the laws of probability, assumed to be real numbers within the [0,1] range. As an agent's degrees of belief in a proposition increase, so does their confidence in its truth.

However, this probabilistic approach may not adequately capture the complexities and subtleties of human belief systems. One issue to consider is that belief and probability are not interchangeable (Spohn, 2013). Although probability distributions are useful, they are fundamentally an additive measure primarily suited for calculating the expected utility of various actions (Smets, 2002). Probabilities are not ideal for representing the full spectrum of human beliefs. The lottery paradox, as described by Kyburg (1961), highlights the difficulties in probabilistic reasoning. Consider a fair lottery with one million tickets, of which only one ticket will win. Each ticket has a one-in-a-million chance of winning – so low that we can be practically certain that any given ticket will not win. However, since one ticket is guaranteed to win, asserting that every ticket will lose leads to a logical inconsistency. This paradox exemplifies a fundamental challenge in probabilistic reasoning – probability theory often struggles with conjunctive inferences, where the additive nature of probabilities can lead to jointly inconsistent conclusions (Wedgwood, 2002). Such scenarios reveal how combining individual probabilistic assessments, even though plausible on their own, can result in an overall belief system that is logically contradictory.

This study introduces ranking theory, developed by Wolfgang Spohn in 1983, as an alternative to traditional probabilistic models. It provides a normative account of belief dynamics and uncertainty representation. It is based on formal epistemology and rationality principles (Skovgaard-Olsen, 2016; Spohn, 2012-2013). Unlike probability-based models, ranking theory does not rely on assigning probabilities to events. It utilizes the set of natural numbers plus infinity to assign degrees of disbelief or belief to propositions, reflecting their logical relations and supporting evidence. Ranking theory addresses the lottery paradox by using ranks to manage beliefs rather than probabilities. It assigns a high-valued rank of disbelief to each ticket based on its low individual probability while accommodating the conditional belief that if a ticket is chosen, it must be the winner. This approach allows for a flexible framework where varying degrees of beliefs can coexist without conflict, simplifying adjustments compared to probabilistic recalculations. These practical implications of ranking theory make it a promising tool for understanding and representing human beliefs and uncertainties.

Despite its advantages, ranking theory faces challenges. It can be complex to understand and apply due to its substantial conceptual shift from traditional probabilistic models. This shift requires a deeper understanding of formal logic and epistemology. Additionally, ranking theory involves managing a broad scale that includes natural number plus infinity, which presents unique challenges in both learning and practical applications. Furthermore, empirical validation is needed to confirm the applicability of this model in human belief modelling as it has not been tested in psychological or behavioural experiments. Lastly, integrating ranking theory with existing probabilistic models poses challenges that must be addressed for broader acceptance and application (Skovgaard-Olsen, 2016).

This study aims to address these challenges by conducting a series of behavioural experiments to examine the empirical adequacy of ranking theory in diverse scenarios and contexts. By exploring the utility of ranking theory as both a normative and descriptive model, this study seeks to advance our understanding of human belief representation, contributing to the development of a model that clarifies the cognitive processes and mechanisms involved.

The basics of ranking theory: a formal explanation

Ranking theory quantifies a grading of *disbelief* expressed by *negative* ranking functions (κ), representing degrees of disbelief about propositions. It utilizes possible world semantics, defining propositions within a comprehensive set, W. An algebra, $\mathcal{A}$, formed from subsets of W, are referred to as *propositions* (Spohn, 2013).

Eq. 1: negative ranking function Let $\mathcal{A}$ be an algebra over W. Then κ, is a negative ranking function for $\mathcal{A}$ iff κ is a function from $\mathcal{A}$ into $\mathbb{R}^* = \mathbb{R}^+ \cup \{\infty\}$ (the set of non-negative reals plus infinity) such that for all $A, B \in \mathcal{A}$:

$$\kappa(W) = 0 \text{ and } \kappa(\emptyset) = \infty$$

If $\kappa(A) = 0,$ A is not disbelieved (not surprising); if $\kappa(A) > 0,$ A is disbelieved (surprising); if $\kappa(A) = \infty$, A is considered impossible.

Eq. 2: law of negation

$$\text{either } \kappa(A) = 0 \text{ or } \kappa(\bar{A}) = 0 \text{ (or both)}$$

The law of negation states $\kappa(A) = 0$ implies A is not disbelieved, allowing $\kappa(\bar{A}) = 0$, where $\bar{A}$ is the negation of A; in such cases, indifference or suspension of judgement regarding A is observed. Importantly, it also asserts that $\kappa(A)$ and $\kappa(\bar{A})$ cannot both be *greater than* 0, ensuring that one cannot simultaneously disbelieve both A and its negation.

Eq. 3: law of disjunction

$$\kappa(A \cup B) = \min\{\kappa(A), \kappa(B)\}$$

Eq. 4: positive ranking function

$$\beta(A) = \kappa(\bar{A})$$

A positive ranking function β expresses degrees of belief for A, defined by the disbelief in the negation of A. If $\beta(A) > 0$, A is believed (to some positive degree); if $\beta(A) = 0$, A is not believed; if $\beta(A) = \infty$, A is believed with absolute certainty.

Eq. 5: two-sided ranking function Let $\mathcal{A}$ be an algebra over W. Then τ is a two-sided ranking function for $\mathcal{A}$ iff τ is a function from $\mathcal{A}$ into $\mathbb{R} \cup \{-\infty, \infty\}$, such that for all $A \in \mathcal{A}$:

$$\tau(A) = \kappa(\bar{A}) - \kappa(A) = \beta(A) - \kappa(A)$$

A two-sided ranking function τ exists for A if and only if there is a negative ranking function κ and a positive counterpart β. Thus, if $\tau(A) > 0$, A is believed; if $\tau(A) < 0$, A is disbelieved (surprising); and if $\tau(A) = 0$, A is neither believed nor disbelieved.

Spohn (2012) suggests that negative ranking functions are more suitable for dynamic belief systems than positive or two-sided ranking functions due to their simplicity and effectiveness. While epistemological studies historically focused on intuitive and manageable degrees of belief (Cohen, 1997; Dubois & Prade, 2023), positive ranking functions are inherently complex, and two-sided ranking functions lack a consistent axiomatic framework. Spohn proposes negative ranking functions, measuring degrees of disbelief, as a clearer and more straightforward method for adjusting beliefs considering new evidence, ensuring a robust and adaptable methodology within the theoretical framework of ranking theory (Spohn, 2012).

Experiment 1 and 2

In Experiment 1, we aim to determine whether humans can express degrees of disbelief (surprise) in terms of negative ranking functions and examine the relationship between negative ranks and subjective probability as a preliminary step toward applying ranking theory in a psychological application. To address these research questions, the first experiment had participants grade degrees of disbelief towards a set of propositions that they may have pre-existing beliefs about the world. Scenarios were given where the objective probability of the given scenarios was unclear. This was to prevent priming participants to think in terms of probability. Experiment 2 was an in-lab replication of Experiment 1 (online).

Participants

A total of 168 undergraduate students (Experiment 1, online) and 63 undergraduate students (Experiment 2, in-lab) from the University of Waterloo participated in the study, receiving course credits in exchange for their involvement. The study did not have any exclusion criteria. In Experiment 1, the median age was 20 years, with 101 females, 66 males, and 1 prefer not to answer. In Experiment 2, conducted during a subsequent semester, the median age was 19 years, with 48 females, 13 males, and 2 undisclosed.

Procedure

Participants assigned numerical values to indicate their degrees of disbelief towards various propositions in a questionnaire (Eq. 1). These values ranged from zero (not surprising) to infinity (impossible), reflecting degrees of disbelief/surprise. The questionnaire included diverse topics with uncertain probabilities, such as language demographics in Quebec, September weather in a specific city, and coworkers' music preferences during a hypothetical karaoke party.

Following this, we employed an open sampling format with a 10x10 grid, where participants filled grid cells based on their disbelief levels from the questionnaire. This method, avoiding explicit probability estimations, aimed to capture a more intuitive expression of disbelief. Participants adjusted the grid to reflect their perceptions, with more cards in a cell indicating a more common occurrence. This approach helps reduce priming effects, facilitating a clearer expression of subjective probabilities (Tiede et al., 2022).

Our investigation explores the relationship between subjective probability and ranking functions, driven by the need to understand how changes in one correspond to changes in the other. This exploration is crucial for understanding the granularity and sensitivity of belief adjustments under ranking theory, where beliefs are commonly represented as subjective probabilities in existing literature. We aim to map subjective probabilities onto a ranking framework, considering Spohn's suggestion that negative ranking functions might closely approximate a logarithmic function with a very small base, enriching our understanding of belief dynamics (Spohn, 2012; Raidl & Skovgaard-Olsen, 2017).

Results

Our analysis predominantly utilized visual methods to illustrate participants' use of negative ranking functions to express disbelief, as shown in Figure 1. This consistent pattern across different propositions was evident in all three questionnaire scenarios. Experiment 2 replicated these findings, reinforcing the consistency of the median rank of 4 observed in both Experiment 1 and 2. For each participant in both experiments, we calculated the range of ranks within each questionnaire. We then determined the median of these rank ranges across all participants and questionnaires. Our analysis revealed a median range of 9 in Experiment 1 and 8 in Experiment 2. These consistent results suggest a stable pattern in the rank variation among participants in both experiments. Figure 2 revealed a logarithmic relationship between subjective probabilities and negative ranks, with a logarithmic base between 0 and 1. In instances where propositions did not evoke disbelief – implying the possible presence of varying belief levels – participants demonstrated a broad spectrum of subjective probabilities, from 0 to 1. The context-dependent nature of subjective probability thresholds for disbelief is particularly emphasized in the purple box in Figure 2, illustrating how participants' responses adapt based on different scenarios.

Discussion

Experiments 1 and 2 demonstrated that participants consistently used negative ranking functions to express disbelief across various propositions and scenarios. The observed logarithmic relationship in Figure 2, which aligns with Spohn's theoretical framework, illustrates the granularity that ranking functions offer in mapping disbelief, providing a clearer distinction than conventional subjective probability assessments. However, the study's exclusive focus on negative ranks limits our exploration of the broader spectrum of belief, particularly for responses that may not inherently suggest disbelief, such as neutral or having degrees of belief toward a proposition. Integrating positive ranking functions in future studies is crucial to expand our understanding of disbelief quantification. A two-sided ranking function would not only increase the granularity of the belief system but also clarify the significance of a rank of 0 within the context of subjective probabilities.

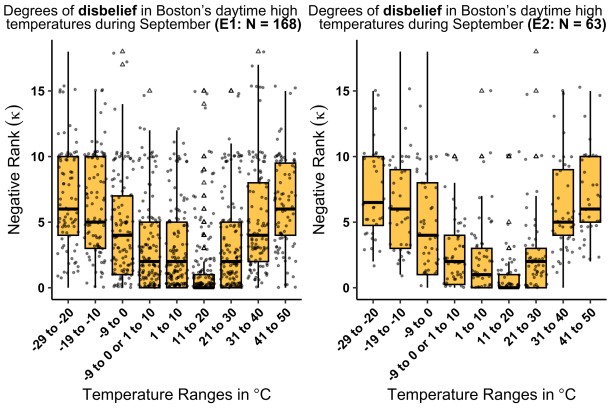

Figure 1: Disbelief in predicted daytime high temperatures for Boston in September. Experiments 1 and 2 demonstrate consistent medians and ranges of disbelief across different temperature propositions.

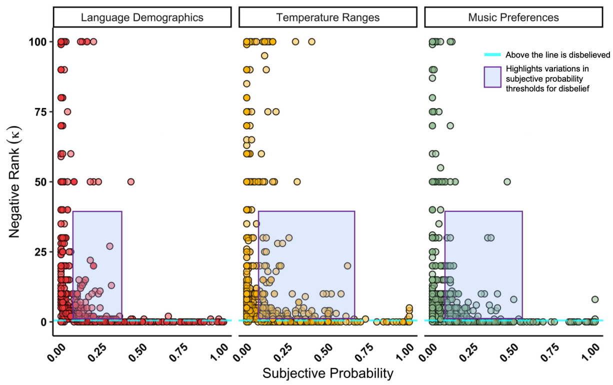

Figure 2: Logarithmic relationship between subjective probability and negative ranks. The purple region illustrates variations in the threshold of subjective probability leading to disbelief across questionnaires. Points below the dotted line represent unsurprising outcomes, with subjective probabilities ranging from 0 to 1.

Experiment 3

Building on the groundwork laid by Experiments 1 and 2, Experiment 3 introduced positive ranks to explore a more comprehensive model of belief and disbelief. This adjustment aimed to enhance the granularity with which subjective probabilities and the two-sided ranking function interrelate, reflecting both ends of the belief spectrum.

Participants

200 undergraduate students from the University of Waterloo participated in the online experiment in exchange for a

course credit. This study has no exclusion criteria (median age 19 years; 156 females; 44 males).

Procedure

The procedure for Experiment 3 mirrored that of Experiments 1 and 2, incorporating two additional scenarios: "real estate investment" and the "lottery ticket" scenarios. Participants were asked to provide positive ranks in addition to the negative ranks used in earlier experiments.

Results and Discussion

The main result of Experiment 3 is presented in Figure 3, which shows a log-odds relationship between subjective probabilities and two-sided ranks. The inclusion of positive ranks in the graph made it easier to see participants' levels of belief and disbelief. A two-sided rank of 0 in Figure 3 indicates a neutral judgment, while values greater than 0 indicate a belief in the proposition. This approach allowed for a more detailed connection between subjective probability and two-sided rank.

The majority of participants showed consistency in applying positive ranks, following the laws of negation (Eq. 2). This confirms the reliability of the method in measuring belief degrees. The two-sided ranking approach effectively assesses belief and disbelief in assessments, as demonstrated by its consistency in both negative and positive ranks.

Experiment 3 contributes to our understanding of ranking theory's application in various contexts and enables us to interpret the complete range of subjective beliefs.

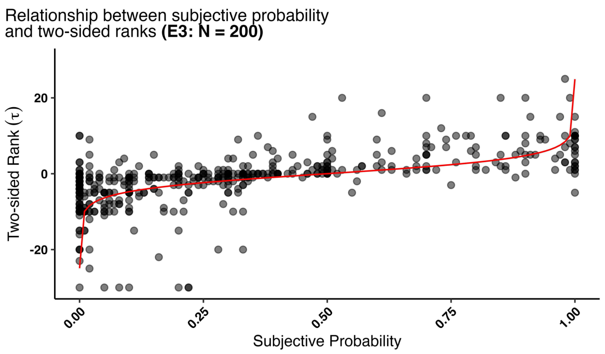

Figure 3: The relationship between subjective probabilities and two-sided ranking functions captures the full spectrum of participant beliefs. The Y-axis scales from disbelief (negative values) to belief (positive values), with zero indicating neutrality.

Experiment 4

Experiment 4 builds on Experiments 1-3, where objective probabilities were unknown, by investigating ranking theory in a learning setting. In this experiment, participants first get acquainted with a computer opponent's hide-and-seek strategy, then provide ranks and subjective probability assessments. The experiment manipulates probability distributions to study their effects on disbelief and belief levels, and their associations with subjective and objective probabilities.

Participants

268 undergraduate students from the University of Waterloo participated in the experiment for course credit, with options for online or in-lab participation. After excluding 67 participants for not adhering to instructions, 201 were included in the final analysis (median age: 20 years; 140 females, 56 males, 5 undisclosed).

Procedure

The experiment began with a practice session where participants alternated between seeking and hiding against a computer opponent, each taking three turns in both roles with randomized room assignments. Following this, the main experiment involved structured trials alternating between seeking and hiding, with each action classified as a 'trial' and every 10 trials comprising a 'round.' Participants completed 7 rounds of each role per game against each opponent, totaling 140 trials per game (70 seeking and 70 hiding).

Each opponent was randomly assigned to one of the probability distributions for room selection ([100, 0, 0], [70, 30, 0], [80, 15, 5], [33.3, 33.3, 33.3]) that they followed for both hiding and seeking, to ensure predictability amidst the overall strategic complexity. During the seeking phase, participants were guided via a dialogue box to find the opponent hiding in one of the rooms, with outcomes indicated by green (success) or red (failure) notifications (i.e., 'you found [child]!' in green). Similarly, during the hiding phase, participants chose a room to hide in, with the feedback displayed through colour-coded notifications.

After each game, participants completed the 'Ranking Questionnaire' and the 'Subjective Probability Questionnaire.' These questionnaires assessed their degrees of disbelief and belief and subjective probability estimates regarding the opponents' likely room choices. Numerical values were assigned to reflect their belief levels or probabilities, offering insights into participants' internal models of opponent behaviour in this dynamic and structured game environment.

Results and Discussion

In an experiment of hide-and-seek game, the participants interacted with computer opponents who had different objective probability distributions as strategies. For every room, the participants gave their degrees of belief/disbelief and subjective probabilities responses.

In Figure 4, we observed the relationship between two-sided ranking functions and objective probability. We began by examining the extreme p-values, and introduced rooms with a p-value of 0 and 1 (i.e., [100-0-0] distribution). A p-value of 0 represented maximum levels of disbelief (denoted by a -infinity value for impossibility), whereas a p-value of 1 represented maximum levels of belief (denoted by a +infinity value for absolute certainty). The violin plot in Figure 4 illustrates that the majority of participants' levels of disbelief and belief matched these extreme p-values.

A two-sided rank of 0 suggests that participants demonstrated indifference in belief between the possibility of the opponent being in a specific room and not being in that room, based on the provided probabilities. This equilibrium in belief systems observed notably in the uniform [33.3, 33.3, 33.3] distribution, indicates that no specific room was perceived as definitively more or less likely for the opponent's presence or absence.

Additionally, our results demonstrated that the two-sided rank increased with the objective probability of the opponent's presence in a particular room, highlighting a positive correlation. This trend was particularly evident in distributions like [80-15-5] and [70-30-0], see Figure 4.

To directly compare ranks and subjective probabilities with objective probabilities, we utilized a 2D probability simplex for visualization in the uniform distribution condition. This analysis revealed that ranks (115 responses) were in agreement with the objective probabilities than subjective probabilities (66 responses in agreement), as illustrated in Figure 5. The plotted points on the simplex, whose sizes were adjusted based on frequency, showed that ranks were much more accurate in representing the objective probabilities than subjective probabilities.

This study reveals discrepancies in how participants process ranks and subjective probabilities, suggesting challenges in estimating probabilities accurately. This invites further research into the cognitive processes involved in decision-making and uncertainty management in various contexts.

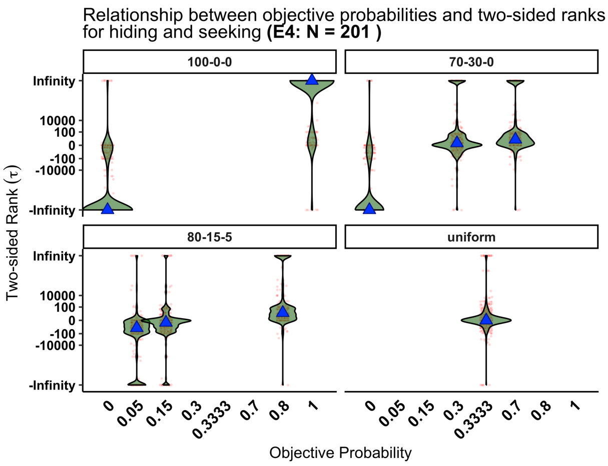

Figure 4: Violin plot illustrating participants' degrees of disbelief and belief for each probability distribution, adjusted using the asinh transformation to scale values from -infinity (representing impossibility) to infinity (representing certainty). This plot compares the relationship between objective probabilities and the two-sided ranks, where negative values indicate disbelief and positive values indicate belief. The triangle within each violin plot represents the median, highlighting how participants' belief systems adapt to different probability distributions.

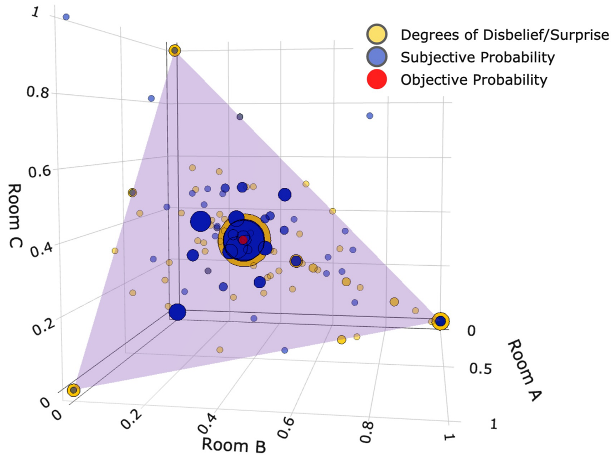

Figure 5: A probability simplex (Δ^{n-1}) displays participants' subjective probabilities and disbelief rankings against a fixed objective probability (1/3, 1/3, 1/3) marked in red. Blue dots, representing subjective probabilities, are widely dispersed, illustrating varied response densities from multiple participants. Yellow dots, indicating negative ranks, cluster at the center, reflecting a more consistent evaluation of objective probability, with each room ranked similarly. Larger dots indicate more common responses, highlighting that rankings provide a clearer picture of the uncertainty structure compared to subjective probabilities.

Experiment 5

Ellsberg's one-urn paradox is a classic problem in decision theory that shows how people violate the expected utility theory when faced with ambiguity, or incomplete information about the probability of different outcomes. In this study, we propose a novel approach to investigate how people assign degrees of belief under uncertainty, employing ranking theory as a framework. In the one-urn problem, we measure individuals' degrees of disbelief prior to making their preferences among bets. We aim to answer the following research questions: (1) Do people exhibit inconsistent beliefs consistent with the Ellsberg paradox, or do our findings challenge the expected inconsistency when using ranks? (2) How does ranking theory explain the influence of ambiguity and belief on decision-making under uncertainty? (3) How does ranking theory resolve the apparent paradox that Ellsberg poses for expected utility theory?

Participants

We recruited 301 undergraduate students from the University of Waterloo for an online experiment in exchange for a course credit. We excluded 10 participants who failed attention checks. (median age: 19 years; 228 females; 53 males; 1 other; 9 prefer not to answer).

Procedure

Participants were presented with an urn scenario involving 90 balls in three colours: red (30 balls), black, and yellow, with the latter two colours totalling 60 but in undisclosed proportions. Participants gave their degrees of disbelief in drawing each colour or combination (i.e., red, black, red or yellow, black or yellow). They then faced two betting scenarios, repeated across four rounds to gauge confidence: betting $100 on drawing (red vs. black), (red or yellow vs. black or yellow), with no reward if another colour was drawn. The four rounds allowed expressions of neutrality by repeating choices. The study included a 'rarity' version with an urn of 1000 balls in four colours: 1 red, 2 black or yellow, and 997 green. Participants assigned degrees of disbelief to this rarity scenario. We maintain the red to black or yellow ratio while increasing rarity to elicit greater degrees of disbelief or surprise, and avoid possible 'floor effect.' An attentiveness check required participants to recall the number of balls and their colours; those failing were excluded from the analysis.

Results

Paradox: Analysis showed significant differences in susceptibility to the Ellsberg paradox. In the initial betting round, 48.8% of participants displayed this susceptibility (choosing "red ball" and "black or yellow"). Over four rounds, the rate dropped to 31.3% (participants who chose similarly at least three times). A chi-squared test indicated a significant decrease in susceptibility with multiple rounds, $X^2(1) = 17.89$, $p < .001$, allowing participants to be neutral about their choice preferences.

With negative ranking functions, only 8.6% displayed susceptibility, decreasing to 5.2% in the rarity scenario (i.e., being less surprised if a 'red' ball is drawn compared to a 'black' ball, and less surprised if a 'black or yellow' ball is drawn compared to a 'red or yellow' ball). A chi-squared test confirmed a significant reduction when using negative rankings compared to betting choices across four rounds, $X^2(1) = 45.49$, $p < .001$. No significant difference was found between one-urn and rarity-urn scenarios in negative rankings, $X^2(1) = 1.85$, $p = .174$.

Disbelief: A two-way ANOVA revealed no difference in disbelief scores across propositions, $F(3, 1996) = .003$, $p = .999$, but a significant effect of urn-scenario, $F(1, 1996) = 5.206$, $p < .05$, with greater disbelief in the rarity scenario (*Mdn* = 97.5) compared to the standard scenario (*Mdn* = 4).

Disbelief and betting choices: Pearson's correlation revealed no significant link between betting choices and disbelief levels in both betting scenarios, thus not rejecting the null hypothesis of zero correlation: $r(276)=0.07$, $p=.264$, 95% CI [−0.05,0.18]; $r(277)=0.07$, $p=.181$, 95% CI [−0.05,0.18].

Discussion

The results of Experiment 5 indicate that expressing beliefs as ranking functions diminishes the impact of the Ellsberg paradox. Only 8.6% of participants exhibited susceptibility to the paradox when using negative ranking functions, compared to binary betting choices. This finding challenges Ellsberg's assertion that gambling choices reflect underlying beliefs, as no significant relationship was observed between participants' disbelief and their betting choices, indicating that choices do not reliably infer beliefs in this context.

To maintain focus on the decision-making variables central to the Ellsberg paradox, subjective probability judgments were not directly measured. This is because subjective probabilities are the only variable that would influence choice behaviour in the Ellsberg paradox. This methodology is consistent with established practices in Ellsberg paradox studies (Oechssler & Roomets, 2015).

Ambiguity aversion – the tendency to prefer bets with known probabilities over those with unknown probabilities, even when the expected values are the same (Ellsberg, 1961) – was less influential under ranking theory. This suggests that ranking theory provides a more objective measure under uncertainty, resistant to the comparative ignorance and competence hypotheses traditionally linked to ambiguity aversion (Fox & Tversky, 1995; Heath & Tversky, 1991). Despite simultaneous assessments of disbelief, participants showed no expected ambiguity aversion in betting, indicating that ranking theory allows a more objective evaluation of uncertainty, unaffected by comparative biases, which typically enhances ambiguity aversion (Fox & Tversky, 1995; Chow & Sarin, 2001).

Moreover, ranking theory might resolve the apparent contradiction posed by the Ellsberg paradox to expected utility theory, which assumes that rational decision-makers assign probabilities to outcomes and choose the option that maximizes expected utility (Schoemaker, 1982). Although the betting scenarios offered identical expected utilities, a significant portion of participants (31.3%) showed inconsistency in their subjective probabilities inferred from their choices. This inconsistency was not present in negative ranking responses, suggesting that ranking theory could offer a more resilient framework for capturing beliefs amid uncertainty and provide a more accurate reflection of decision-making processes when probabilities are ambiguous.

Conclusion

Our series of five experiments explored the application of ranking functions to the expression of belief and disbelief. Experiments 1 and 2 demonstrated that negative ranking functions effectively map disbelief, offering finer granularity than traditional subjective probabilities. Experiment 3 introduced positive ranks, affirming the utility of two-sided ranking functions for capturing a full spectrum of beliefs. Experiment 4 further validated this approach in a hide-and-seek game scenario, where ranking functions more accurately reflected objective probabilities than subjective probabilities. Lastly, Experiment 5 addressed the Ellsberg paradox, showing a diminished impact of ambiguity aversion when beliefs were quantified using ranking functions, thus challenging conventional interpretations of decision-making under uncertainty. These findings emphasize the ability of ranking theory to capture complex belief dynamics and enhance our understanding of cognitive processes in decision-making contexts.

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